Exam Problem 2c: $\alpha = \inf A$, $\exists \{a_n\} \subseteq A$, $\lim_{n\to\infty} a_n = \alpha$.

Common Mistake: $\forall n$, $\exists a_n \in A$, $a_n = \alpha + \frac{1}{n}$.

Counter Example: $A = \{ \frac{1}{n+\sqrt{2}} : n = 1, 2, \cdots \}$. $\inf A = 0$

There doesn't exist any $a \in A$ s.t. $a = \frac{1}{n}$

b/c all denominators in A are irrational !!

Correct: $\forall n$, $\exists a_n \in A$, s.t. $a_n < \alpha + \frac{1}{n}$.

Sol'n: $\alpha = \inf A \Rightarrow \forall \varepsilon > 0$, $\exists a \in A$, s.t. $a < \alpha + \varepsilon$

($\forall \varepsilon > 0$, $\alpha + \varepsilon$ is not a lower bound)

For $\varepsilon = 1$, $\exists a_1 \in A$, $a_1 < \alpha + 1$

$\varepsilon = \frac{1}{2}$, $\exists a_2 \in A$, $a_2 < \alpha + \frac{1}{2}$

$\vdots \qquad \vdots \qquad \vdots$

$\varepsilon = \frac{1}{n}$, $\exists a_n \in A$, $a_n < \alpha + \frac{1}{n}$.

Also notice that $a_n \in A$. $\Rightarrow a_n \geq \alpha$.

$\Rightarrow \forall n \in \mathbb{N}$, $\alpha \leq a_n < \alpha + \frac{1}{n}$

Squeeze Lemma $\Rightarrow \lim_{n\to\infty} a_n = \alpha$

Problem 3b. (i) $f(x) = \frac{x}{x-1}$ does not have a limit at $x = 1$.

$= 1 + \frac{1}{x-1}$.

Pick $x_n = 1 + \frac{1}{n}$. $f(x_n) = 1 + \frac{1}{1+\frac{1}{n}-1} = 1 + n$.

As $n \to \infty$, $f(x_n) \to \infty$. In particular, $\{f(x_n)\}$ is unbounded.

So $\{f(x_n)\}$ diverges. $\Rightarrow \lim_{n\to\infty} f(x_n)$ DNE $\Rightarrow \lim_{x\to 1} f(x)$ DNE

Recall: Function Limits in terms of sequence.

$$\lim_{x\to c} f(x) = L \iff \forall \{x_n\} \subseteq D,\ x_n \to c,\ x_n \neq c \Rightarrow f(x_n) \to L$$

Argue by Cauchy's Criterion:

Recall: $\lim_{x\to c} f(x)$ exists $\iff \forall \varepsilon>0, \exists \delta>0, \forall x,y \in (c-\delta, c+\delta)\setminus\{x_0\}$.

$$|f(x)-f(y)|<\varepsilon.$$

Rmk: Cauchy's criterion is handy when we have no information about the limit

Need to show: $\exists \varepsilon>0, \forall \delta>0, \exists x,y \in (1-\delta, 1+\delta)\setminus\{1\}$.

$$|f(x)-f(y)| \geq \varepsilon.$$

Pick $\varepsilon = 1$, $\forall \delta>0$, Pick $x = 1+\frac{1}{n}$, $y = 1+\frac{1}{n+1}$.

where n is an integer, s.t. $\frac{1}{n}<\delta$.

$$\frac{1}{n}<\delta \Rightarrow 1+\frac{1}{n} < 1+\delta \Rightarrow x \in (1-\delta, 1+\delta)\setminus\{1\}$$

$$\frac{1}{n+1}<\frac{1}{n}<\delta \Rightarrow y = 1+\frac{1}{n+1} \in (1-\delta, 1+\delta)\setminus\{1\}$$

$$|f(x)-f(y)| = \left|1+\frac{1}{x-1} - \left(1+\frac{1}{y-1}\right)\right|$$

$$= \left|\frac{1}{1+\frac{1}{n}-1} - \frac{1}{1+\frac{1}{n+1}-1}\right|$$

$$= |n-(n+1)| = 1 \geq \varepsilon.$$

Digression: Why $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Recall: $\sum_{n=1}^{\infty} a_n$ converges $\iff$ The sequence $S_n = \sum_{k=1}^{n} a_k$ converges.

Recall: $\{a_n\}$ converges $\iff \forall \varepsilon>0, \exists N \in \mathbb{Z}_+, \forall m,n>N, |a_m - a_n| < \varepsilon$.

Need to see what $S_n - S_m$ is $(n > m)$

$$S_n - S_m = \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{n^2}$$

Notice: $\frac{1}{k^2} = \frac{1}{k} \cdot \frac{1}{k} < \frac{1}{k} \cdot \frac{1}{k-1} = \frac{1}{k-1} - \frac{1}{k}$.

$$S_n - S_m < \frac{1}{m} - \cancel{\frac{1}{m+1}} + \cancel{\frac{1}{m+1}} - \cancel{\frac{1}{m+2}} + \cancel{\frac{1}{m+2}} - \cancel{\frac{1}{m+3}} + \cdots + \cancel{\frac{1}{n-1}} - \frac{1}{n}.$$

TELESCOPING SUM

$$= \frac{1}{m} - \frac{1}{n}.$$

For every $\varepsilon > 0$, pick $N > \frac{1}{\varepsilon}$. then $\forall m, n > N$ with $n > m$.

$$|S_n - S_m| = \frac{1}{m} - \frac{1}{n} < \frac{1}{m} < \frac{1}{N} < \varepsilon$$

$\Rightarrow \{S_n\}$ converges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Problem 3b. ii. $f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x & x \in \mathbb{Q}. \end{cases}$ does not have a limit at any $c \neq 0$
has a limit at $c = 0$.

For $c \neq 0$. Use sequential criterion to show $\lim_{x \to c} f(x)$ DNE.

Pick $\{r_n\} \subseteq \mathbb{Q}$, $r_n \to c$, $r_n \neq c$. Pick $\{s_n\} \subseteq \mathbb{R} \setminus \mathbb{Q}$, $s_n \to c$, $s_n \neq c$

Why can I choose such a sequence?

Recall: Density Thm of Rat'ls: $\forall a, b \in \mathbb{R}$, $a < b$, $\exists r \in \mathbb{Q}$, $r \in (a, b)$

Density Thm of Irrat'ls: $\forall a, b \in \mathbb{R}$, $a < b$, $\exists s \in \mathbb{R} \setminus \mathbb{Q}$, $s \in (a, b)$

Proofs available in both my notes and "Understanding Analysis".

For $c \in \mathbb{R}$, $n \in \mathbb{Z}_+$, consider the interval $(c - \frac{1}{n}, c + \frac{1}{n})$

Density Thm $\Rightarrow \exists r_n \in \mathbb{Q}$, $r_n \in (c - \frac{1}{n}, c + \frac{1}{n})$.

$s_n \in \mathbb{R} \setminus \mathbb{Q}$, $s_n \in (c - \frac{1}{n}, c + \frac{1}{n})$.

For $n=1$, $\Rightarrow r_1 \in \mathbb{Q}$, $s_1 \notin \mathbb{Q}$, $r_1 \in (c-1, c+1)$
$s_1 \in (c-1, c+1)$

For $n=2$, $\Rightarrow r_2 \in \mathbb{Q}$, $s_2 \notin \mathbb{Q}$, $r_2 \in (c-\frac{1}{2}, c+\frac{1}{2})$
$s_2 \in (c-\frac{1}{2}, c+\frac{1}{2})$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

For $n \in \mathbb{Z}_+$. $\Rightarrow r_n \in \mathbb{Q}$, $s_n \notin \mathbb{Q}$, $r_n \in (c-\frac{1}{n}, c+\frac{1}{n})$
$s_n \in (c-\frac{1}{n}, c+\frac{1}{n})$

$\Rightarrow$ sequences $\{r_n\}$ in $\mathbb{Q}$, $\{s_n\}$ in $\mathbb{R}\setminus\mathbb{Q}$.

$$r_n \to c, \quad s_n \to c.$$

But $c \neq 0$. $f(r_n) = r_n \to c \neq 0$
$f(s_n) = 0 \to 0$

So two seq. $\{r_n\}$, $\{s_n\} \to c$, w/ $\lim_{n\to\infty} f(r_n) \neq \lim_{n\to\infty} f(s_n)$.

$$\Rightarrow \lim_{x\to c} f(x) \quad \text{DNE}.$$

<u>Exercise</u>: Try to argue by Cauchy's criterion.

Hint: ε can be taken to be a multiple of c, say $\frac{1}{2}c$.

$\forall \delta > 0$, $\exists\, r \in \mathbb{Q}$, $s \notin \mathbb{Q}$, $r, s \in (c-\delta, c+\delta)$

$$|f(r) - f(s)| = |r - 0| > \max(|c-\delta|, |c+\delta|)$$
$$\geq \frac{1}{2}c \quad \text{if } \delta < \frac{1}{2}c.$$

$c = 0$. $\lim_{x\to 0} f(x)$ exists. and is 0.

$\forall \varepsilon > 0$, pick $\delta = \frac{1}{2}\varepsilon$. then for any $x \in (0-\delta, 0+\delta) \setminus \{0\}$.

if $x \notin \mathbb{R} \setminus \mathbb{Q}$, $f(x) = 0 \Rightarrow |f(x) - 0| < \varepsilon$

$x \in \mathbb{Q}$. $f(x) = x \Rightarrow |f(x) - 0| = |x| < \delta = \frac{1}{2}\varepsilon < \varepsilon$.

$\Rightarrow \lim_{x \to 0} f(x) = 0$ (by definition).

Problem 3b.(iii) $f(x) = x\left(\sin\frac{1}{x}\right)^2$, $\lim_{x \to 0} f(x) = 0$.

Recall: If $\lim_{x \to c} f(x) = 0$, $g(x)$ is bounded near c.

$(\exists M \geq 0,\ \exists \delta > 0,$ s.t. $|g(x)| \leq M,$

$\forall x \in (c-\delta, c+\delta))$

Then $\lim_{x \to c} f(x)g(x) = 0$.

$x \to 0$, $\left(\sin\frac{1}{x}\right)^2 \leq 1$. $\forall x$ near 0.

$\Rightarrow \lim_{x \to 0} x\left(\sin\frac{1}{x}\right)^2 = 0$.

Exercise: Prove the statement in purple by yourself.

Example: $f(x) = \frac{2x}{3x-1}$, Verify by definition that $\lim_{x \to 1} f(x) = 1$.

$$f(x) - 1 = \frac{2x}{3x-1} - 1 = \frac{2x - 3x + 1}{3x - 1} = \frac{-x+1}{3x-1}$$

$$|f(x) - 1| = \frac{|x-1|}{|3x-1|}$$

Since $x \to 1$. $3x - 1 \to 2$. i.e., I can find some δ, s.t.

$x \in (1-\delta, 1+\delta) \setminus \{1\}$, $3x - 1 \geq 1$

δ can be chosen $\frac{1}{3}$.

For $\delta_1 = \frac{1}{3}$, $\forall x \in (1-\delta_1, 1+\delta_1) \setminus \{1\}$.

$$3x - 1 > 3\cdot(1-\delta_1) - 1 = 3\cdot(1-\tfrac{1}{3}) - 1 = 1$$

$$\Rightarrow |f(x) - 1| = \frac{|x-1|}{3x-1} < \frac{|x-1|}{1}$$

If $\delta_2 < \varepsilon$, then $x \in (1-\delta_2, 1+\delta_2) \setminus \{1\}$. $|x-1| < \varepsilon$.

So for $\delta = \min(\delta_1, \delta_2)$. both inequalities hold. i.e. $x \in (1-\delta, 1+\delta) \setminus \{1\}$

$$|f(x) - 1| = \frac{|x-1|}{|3x-1|} = \frac{|x-1|}{3x-1} < \frac{|x-1|}{1} < \varepsilon.$$

<u>Rmk</u>: Inequalities are central to analysis.